Department of Mathematics & Philosophy of Engineering

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Course: MPZ 3231-Engineering Mathematics IA **Model Answer No·01**

Academic Year – 2014/2015

1)

a) $\overrightarrow{OA} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ $\quad \overrightarrow{OB} = 4\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ $\quad \overrightarrow{OC} = 3\boldsymbol{i} + 3\boldsymbol{k}$

i. $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{OA}.\overrightarrow{AB} = (2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}).(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= (2 \times 2) + (-2 \times 1) + (1 \times -2) = 0$

$\overrightarrow{OA}.\overrightarrow{AC} = (2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

$= (2 \times 1) + (-2 \times 2) + (1 \times 2) = 0$

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ both perpendicular to $\overrightarrow{OA}$

ii. $\overrightarrow{OB}.\overrightarrow{OC} = |\overrightarrow{OB}|\,|\overrightarrow{OC}| \cos\theta$

Where θ is the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$

$(4\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}).(3\boldsymbol{i} + 3\boldsymbol{k}) = \sqrt{18}.\sqrt{18}\cos\theta$; $\quad |\overrightarrow{OB}| = |\overrightarrow{OC}| = \sqrt{18}$

$12 - 0 - 3 = 18\cos\theta$

$\cos\theta = \dfrac{1}{2}$

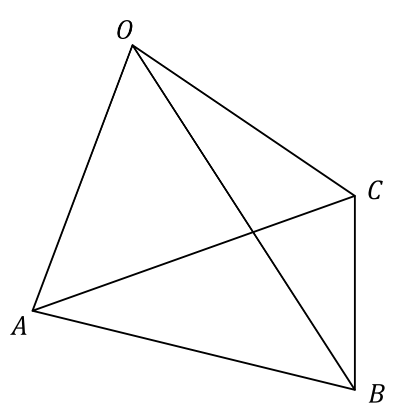
$\theta = 60^0$

iii. $\dfrac{1}{2} = |\overrightarrow{AB} \times \overrightarrow{BC}|$ = area of the triangle ABC

$$|\overrightarrow{AB} \times \overrightarrow{BC}| = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix} = |6\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k}| = \sqrt{81} = 9$$

Area of the triangle ABC $= \frac{1}{2} \times 9$ square units

b)



Given that $OABC$ is a tetrahedron and OA perpendicular to BC and OC perpendicular to AB.

To prove: - OB perpendicular to AC

Proof:- Let the position vectors of A, B, C are $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ respectively with respect to O

$$\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}, \overrightarrow{OC} = \boldsymbol{c}, \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}, \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}, \overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$$

OA perpendicular to BC

$$\overrightarrow{OA}.\overrightarrow{BC} = 0$$
$$\boldsymbol{a}.(\boldsymbol{c} - \boldsymbol{b}) = 0$$
$$\boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{b} = 0 - - - (1)$$

OC perpendicular to AB

$$\overrightarrow{OC}.\overrightarrow{AB} = 0$$
$$\boldsymbol{c}.(\boldsymbol{b} - \boldsymbol{a}) = 0$$
$$\boldsymbol{c}.\boldsymbol{b} - \boldsymbol{c}.\boldsymbol{a} = 0 - - - (2)$$

$(1) + (2)$

$$\boldsymbol{c}.\boldsymbol{b} - \boldsymbol{a}.\boldsymbol{b} = 0 \qquad \text{since } \boldsymbol{a}.\boldsymbol{c} = \boldsymbol{c}.\boldsymbol{a}$$
$$\boldsymbol{b}.(\boldsymbol{c} - \boldsymbol{a}) = 0$$
$$\overrightarrow{OB}.\overrightarrow{AC} = 0$$

$\therefore$ OB perpendicular to AC

c) Let

$$\overrightarrow{OA} = -3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$$
$$\overrightarrow{OB} = -6\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$$
$$\overrightarrow{OC} = -5\boldsymbol{i} - 7\boldsymbol{j} - 3\boldsymbol{k}$$
$$\overrightarrow{OD} = -2\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$$

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -(-3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}) + (-6\boldsymbol{i} - 3\boldsymbol{j} - 1\boldsymbol{k}) = -3\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}$$
$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -(-3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}) + (-5\boldsymbol{i} - 7\boldsymbol{j} - 3\boldsymbol{k}) = -2\boldsymbol{i} - 9\boldsymbol{j} + \boldsymbol{k}$$
$$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = -(-3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}) + (-2\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}) = \boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & -5 & 3 \\ -2 & -9 & 1 \end{vmatrix}$$
$$= (-5 + 27)\boldsymbol{i} + (-6 + 3)\boldsymbol{j} + (27 - 10)\boldsymbol{k}$$
$$= 22\boldsymbol{i} - 3\boldsymbol{j} + 17\boldsymbol{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC}.\overrightarrow{AD} = (22\boldsymbol{i} - 3\boldsymbol{j} + 17\boldsymbol{k})(i - 4j - 2k)$$
$$= 22 + 12 - 34$$
$$= 0$$

or

$$\overrightarrow{AB} \times \overrightarrow{AC}.\overrightarrow{AD} = \begin{vmatrix} -3 & -5 & 3 \\ -2 & -9 & 1 \\ 1 & -4 & -2 \end{vmatrix} = 0$$

Hence the given four points are coplanar.

Alternative method

If A, B, C, D are coplanar
then $AB \; // \; CD$ or AB and CD are intersect

If $AB // \; CD$;
then $\overrightarrow{AB} = \lambda \overrightarrow{CD}$

$$\overrightarrow{AB} = -3\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}$$
$$\begin{aligned} \overrightarrow{CD} &= \overrightarrow{CO} + \overrightarrow{OD} \\ &= -(-5\boldsymbol{i} - 7\boldsymbol{j} - 3\boldsymbol{k}) + (-2\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}) \\ &= 3\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k} \end{aligned}$$

$\therefore AB = DC$
$\therefore AB // DC$

$\therefore A, B, C, D$ are coplanar.

2.

a) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0 - - - (1)$

$(1) \times \boldsymbol{a}$
$$\begin{aligned} (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a} &= \boldsymbol{0} \times \boldsymbol{a} \\ \boldsymbol{a} \times \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a} &= \boldsymbol{0} \\ \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a} &= \boldsymbol{0} \\ \boldsymbol{c} \times \boldsymbol{a} &= -\boldsymbol{b} \times \boldsymbol{a} \\ \boldsymbol{c} \times \boldsymbol{a} &= \quad \boldsymbol{a} \times \boldsymbol{b} - - - (2) \end{aligned}$$

$(1) \times \boldsymbol{b}$
$$\begin{aligned} (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{b} &= \boldsymbol{0} \times \boldsymbol{b} \\ \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{b} + \boldsymbol{c} \times \boldsymbol{b} &= \boldsymbol{0} \\ \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{c} \times \boldsymbol{b} &= \boldsymbol{0} \\ \boldsymbol{a} \times \boldsymbol{b} &= -\boldsymbol{c} \times \boldsymbol{b} \\ \boldsymbol{a} \times \boldsymbol{b} &= \quad \boldsymbol{b} \times \boldsymbol{c} - - - (3) \end{aligned}$$

From (2) and (3) $\quad \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$

b)

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c} - - - (1)$$
$$\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) = (\boldsymbol{b}.\boldsymbol{a})\boldsymbol{c} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a} - - - (2)$$
$$\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{c}.\boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c}.\boldsymbol{a})\boldsymbol{b} - - - (3)$$

$(1) + (2) + (3)$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b})$$
$$= (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b}.\boldsymbol{a})\boldsymbol{c} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a} + (\boldsymbol{c}.\boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c}.\boldsymbol{a})\boldsymbol{b}$$
$$= \boldsymbol{0}$$

c) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0} - - - (1)$
$|\boldsymbol{a}| = 5, \quad |\boldsymbol{b}| = 7, \quad |\boldsymbol{c}| = 3$

$(1).\boldsymbol{a}$
$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}).\boldsymbol{a} = \boldsymbol{0}.\boldsymbol{a}$
$\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{a} + \boldsymbol{c}.\boldsymbol{a} = 0$

$|\boldsymbol{a}|^2 + |\boldsymbol{b}||\boldsymbol{a}|\cos\theta + |\boldsymbol{c}||\boldsymbol{a}|\cos\varphi = 0$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$
where φ is the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$
$|\boldsymbol{a}| \neq 0$

$|\boldsymbol{a}| + |\boldsymbol{b}|\cos\theta + |\boldsymbol{c}|\cos\varphi = 0$
$5 + 7\cos\theta + 3\cos\varphi = 0 - - - (2)$

$(1).\boldsymbol{b}$
$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}).\boldsymbol{b} = \boldsymbol{0}.\boldsymbol{b}$
$\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{b} = 0$

$|\boldsymbol{a}||\boldsymbol{b}|\cos\theta + |\boldsymbol{b}|^2 + |\boldsymbol{c}||\boldsymbol{b}|\cos\gamma = 0$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$
where γ is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$
$|\boldsymbol{b}| \neq 0$

$|\boldsymbol{a}|\cos\theta + |\boldsymbol{b}| + |\boldsymbol{c}|\cos\gamma = 0$
$5\cos\theta + 7 + 3\cos\gamma = 0 - - - (3)$

$(1).\boldsymbol{c}$
$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}).\boldsymbol{c} = \boldsymbol{0}.\boldsymbol{c}$
$\boldsymbol{a}.\boldsymbol{c} + \boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c} = 0$

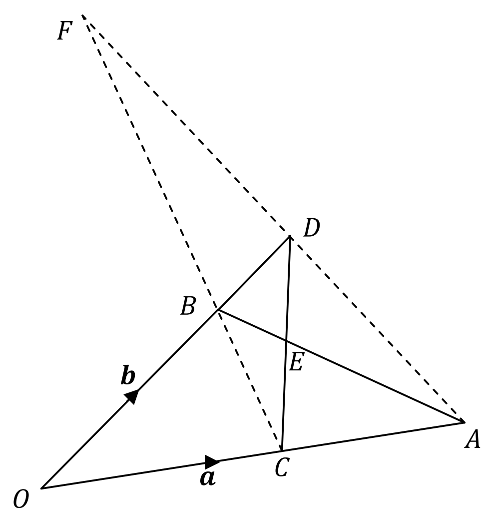
$|\boldsymbol{a}||\boldsymbol{c}|\cos\varphi + |\boldsymbol{b}||\boldsymbol{c}|\cos\gamma + |\boldsymbol{c}|^2 = 0$

$|\boldsymbol{c}| \neq 0$

$|\boldsymbol{a}|\cos\varphi + |\boldsymbol{b}|\cos\gamma + |\boldsymbol{c}| = 0$

$5\cos\varphi + 7\cos\gamma + 3 \ = 0 - - - (4)$
$(3) \times 7 - (4) \times 3$
$35\cos\theta - 15\cos\varphi + 49 - 9 = 0$
$35\cos\theta - 15\cos\varphi + 40 = 0$
$7\cos\theta - 3\cos\varphi + 8 = 0 - - - (5)$
$(2) - (5)$
$6\cos\varphi + 5 - 8 = 0$
$\cos\varphi = \frac{1}{2}$
$\therefore \varphi = 60^0$
Angle between $\boldsymbol{c}$ and $\boldsymbol{a}$ is 60^0

d)



$\overrightarrow{OD} = \mu\boldsymbol{b} \quad \overrightarrow{OC} = \lambda\boldsymbol{a}$
$\overrightarrow{AE} = \alpha\overrightarrow{AB}$
$\overrightarrow{AO} + \overrightarrow{OE} = \alpha\left(\overrightarrow{AO} + \overrightarrow{OB}\right)$
$\overrightarrow{OE} = \alpha(-\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{a}$
$\overrightarrow{OE} = (1-\alpha)\boldsymbol{a} + \alpha\boldsymbol{b} - - - (1)$
$\frac{AE}{AB} = \alpha \qquad 0 < \alpha < 1$
Since $\overrightarrow{OC} = \lambda\boldsymbol{a}$ and $\overrightarrow{OD} = \mu\boldsymbol{b}$
$\therefore \overrightarrow{OE} = (1-\beta)\lambda\boldsymbol{a} + \beta\mu\boldsymbol{b} - - - (2)$
From (1) & (2)
$(1-\alpha) = (1-\beta)\lambda - - - (3)$
$\alpha = \beta\mu - - - (4)$
$(1-\beta\mu) = (1-\beta)\lambda$
$\beta(\lambda-\mu) = (\lambda-1)$
$\beta = \frac{(\lambda-1)}{(\lambda-\mu)}$

$\therefore$ from (2)

$$\overrightarrow{OE} = \left(1 - \frac{(\lambda-1)}{(\lambda-\mu)}\right)\lambda\boldsymbol{a} + \frac{(\lambda-1)}{(\lambda-\mu)}\mu\boldsymbol{b}$$

$$(\lambda-\mu)\overrightarrow{OE} = \big((\lambda-\mu)-(\lambda-1)\big)\lambda\boldsymbol{a} + (\lambda-1)\mu\boldsymbol{b}$$

$$(\lambda-\mu)\overrightarrow{OE} = (1-\mu)\lambda\boldsymbol{a} + (\lambda-1)\mu\boldsymbol{b}$$

$$(\mu-\lambda)\boldsymbol{e} = (\mu-1)\lambda\boldsymbol{a} + (1-\lambda)\mu\boldsymbol{b}$$

Where $\overrightarrow{OE} = \boldsymbol{e}$

We have from (4)

$$\alpha = \beta\mu = \mu\frac{(\lambda-1)}{(\lambda-\mu)}$$

i. If E is the midpoint of AB

$$\alpha = \frac{1}{2}$$

$$\alpha = \frac{1}{2} = \mu\frac{(\lambda-1)}{(\lambda-\mu)}$$

$$\lambda-\mu = 2\mu\lambda - 2\mu$$

$$\lambda+\mu = 2\mu\lambda$$

$$\frac{1}{\lambda}+\frac{1}{\mu} = 2$$

ii. If E is the midpoint of CD then

$$\beta = \frac{1}{2} = \frac{(\lambda-1)}{(\lambda-\mu)}$$

$$\lambda-\mu = 2\lambda - 2$$

$$\lambda+\mu = 2$$

iii. Let $\frac{CF}{CB} = \alpha'$ and $\frac{AF}{AD} = \beta'$

From (1) $\overrightarrow{OF} = (1-\alpha')\lambda\boldsymbol{a} + \alpha'\boldsymbol{b} --- (5)$

From (2) $\overrightarrow{OF} = (1-\beta')\boldsymbol{a} + \beta'\mu\boldsymbol{b} --- (6)$

From (5) and (6)

$$(1-\alpha')\lambda = 1-\beta' \qquad \alpha' = \beta'\mu$$

$$(1-\beta'\mu)\lambda = 1-\beta'$$

$$\lambda - 1 = \beta'(\lambda\mu-1)$$

$$\beta' = \frac{\lambda-1}{\lambda\mu-1}$$

From (6)

$$\overrightarrow{OF} = \left(1-\frac{\lambda-1}{\lambda\mu-1}\right)\boldsymbol{a} + \left(\frac{\lambda-1}{\lambda\mu-1}\right)\mu\boldsymbol{b}$$

$$(\lambda\mu-1)\overrightarrow{OF} = (\lambda\mu-\lambda)\boldsymbol{a} + (\lambda-1)\mu\boldsymbol{b}$$

$$(\lambda\mu - 1)\overrightarrow{OF} = \lambda(\mu - 1)\boldsymbol{a} + \mu(\lambda - 1)\boldsymbol{b}$$
$$(\lambda\mu - 1)\boldsymbol{f} = \lambda(\mu - 1)\boldsymbol{a} + \mu(\lambda - 1)\boldsymbol{b}$$

If $\lambda\mu \neq 1$ the point F does not exist.

3.

a) $f(\lambda) = \lambda^2 - 2\lambda + 2$

$$A = \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}$$

$$A^2 = \underline{A}.\underline{A} = \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -10 \\ 2 & 4 \end{pmatrix}$$

$$f(A) = A^2 - 2A + 2I$$

$$= \begin{pmatrix} -4 & -10 \\ 2 & 4 \end{pmatrix} - 2\begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore f(A) = A^2 - 2A + 2I = 0$$

$$f(1+i) = (1+i)^2 - 2(1+i) + 2$$

$$= 1 + 2i - 1 - 2 - 2i + 2$$

$$= 0$$

$$\therefore f(1+i) = 0$$

$$(1+i)^6 = 1^6 + 6i - 15 - 20i + 15 + 6i - 1$$

$$(1+i)^6 = -8i$$

$$\lambda^6 = f(\lambda)g(\lambda) + \alpha\lambda + \beta$$

When $\lambda = (1+i)$

$$(1+i)^6 = f(1+i)g(1+i) + \alpha(1+i) + \beta$$

$$-8i = \alpha i + \alpha + \beta$$

$\alpha = -8 \qquad \alpha + \beta = 0$

$\beta = 8$

$$\lambda^6 = f(\lambda)g(\lambda) - 8\lambda + 8$$

When $\lambda = \underline{A}$

$$A^6 = f(A)g(A) - 8A + 8I$$

$$A^6 = 0 - 8A + 8I \qquad\qquad f(A) = 0$$

$$= 0 - 8\begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 40 \\ -8 & 16 \end{pmatrix}$$

$$A^6 = 8\begin{pmatrix} 2 & 5 \\ -1 & 2 \end{pmatrix}$$

b) $x + y + z = 6$

$$x - y + 2z = 5$$

$$3x + y + z = 8$$

$$2x - 2y + 3z = 7$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

$$\therefore AX = B$$

$$A, b = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ -3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix} \xleftarrow[R_3 \to {R_3}/{3}]{} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\downarrow R_4 \to R_4 - R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A, b = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ We can say that Rank $[A, b] = 3$, Rank $[A]$=3

We can rearrange the above

$$x + y + z = 6$$

$$-2y + z = -1$$

$$-z = -3$$

$\therefore$ by using backward substitutions

$$z = 3$$

$$y = 2$$

$$x = 1$$

c). $x - 4y + 5z = 8$

$$3x + 7y - z = 3$$

$$x + 15y - 11z = 14$$

$$\begin{bmatrix} 1 & -4 & 5 \\ 3 & 7 & -1 \\ 1 & 15 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 14 \end{bmatrix}$$

$$AX = B \quad ; A = \begin{bmatrix} 1 & -4 & 5 \\ 3 & 7 & -1 \\ 1 & 15 & -11 \end{bmatrix}$$

$$\det A = 1(-77 + 15) + 4(-33 + 1) + 5(45 - 7)$$

$$= -62 - 128 + 190 = 0$$

$$\det A = 0 \quad \begin{vmatrix} 1 & -4 \\ 3 & 7 \end{vmatrix} \neq 0$$

$\therefore$ Rank $A = 2$

$$A, b = \begin{bmatrix} 1 & -4 & 5 & 8 \\ 3 & 7 & -1 & 3 \\ 1 & 15 & -11 & 14 \end{bmatrix}$$

We have $\begin{vmatrix} -4 & 5 & 8 \\ 7 & -1 & 3 \\ 15 & -11 & 14 \end{vmatrix} = -4(-14 + 33) - 5(98 - 45) + 8(-77 + 15) \neq 0$

$\therefore$ Rank $(A, b) = 3$

$\therefore$ Rank (A, b) >Rank A

$\therefore$ The equation are inconsistent and that there exists no solutions

4.

a) $x + 2y - 3z - 4u = 6$

$x + 3y + z - 2u = 4$

$2x + 5y - 2z - 5u = 10$

$$\begin{bmatrix} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & -2 \\ 2 & 5 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & -2 \\ 2 & 5 & -2 & -5 \end{bmatrix}$$

$$A, b = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -11 & 0 & 10` \\ 0 & 1 & 4 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xleftarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -11 & -8 & 10` \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

∴We can say that Rank $(A, b) = 3$

Rank $A = 3$

The equations are consistent.

∴We have

$x - 11z = 10$

$y + 4z = -2$

$u = 0$

$\therefore x = 10 + 11z$

$y = -(2 + 4z)$

$u = 0$

thus the infinite number of solutions. z is arbitrary.

b) $x_1 + 3x_2 - 2x_3 = 0$

$2x_1 - x_2 + 4x_3 = 0$

$x_1 - 11x_2 + 14x_3 = 0$

$$\therefore \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$\det A = 1(-14+44) + 3(4-28) - 2(-22+1)$$

$$= 30 - 72 + 42$$

$$= 0$$

$\det A = 0$

$\therefore$ rank $A = 2$ rank $A <$ number of unknowns

$\therefore$ the given system will have an infinite number of solutions.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 1 & -14 & 16 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The equivalent system

$$x_1 + 3x_2 - 2x_3 = 0$$

$$-7x_2 + 8x_3 = 0$$

$$\therefore \quad x_2 = \frac{8}{7}x_3$$

$$x_1 = 2x_3 - \frac{24}{7}x_3 = -\frac{10}{7}x_3$$

$$x_1 = -\frac{10}{7}x_3 \qquad x_2 = \frac{8}{7}x_3$$

Where x_3 is arbitrary.

d. $$\begin{vmatrix} a^2 & 1-a^2 & 1+a^3 \\ b^2 & 1-b^2 & 1+b^3 \\ x^2 & 1-x^2 & 1+x^3 \end{vmatrix} \xrightarrow[C_2 \to C_1 + C_2]{} \begin{vmatrix} a^2 & 1 & 1+a^3 \\ b^2 & 1 & 1+b^3 \\ x^2 & 1 & 1+x^3 \end{vmatrix} \xrightarrow[C_3 \to C_3 - C_2]{} \begin{vmatrix} a^2 & 1 & a^3 \\ b^2 & 1 & b^3 \\ x^2 & 1 & x^3 \end{vmatrix}$$

$$\downarrow \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$(a-x)(b-x) \begin{vmatrix} a+x & 0 & a^2+ax+x^2 \\ b+x & 0 & b^2+bx+x^2 \\ x^2 & 1 & x^3 \end{vmatrix} \longleftarrow \begin{vmatrix} a^2-x^2 & 0 & a^3-x^3 \\ b^2-x^2 & 0 & b^3-x^3 \\ x^2 & 1 & x^3 \end{vmatrix}$$

$$\downarrow R_1 \to R_1 - R_2$$

$$(a-x)(b-x) \begin{vmatrix} a-b & 0 & (a^2-b^2)+(a-b)x \\ b+x & 0 & b^2+bx+x^2 \\ x^2 & 1 & x^3 \end{vmatrix}$$

$$\downarrow$$

$$(a-x)(b-x)(a-b)\begin{vmatrix} 1 & 0 & a+b+x \\ b+x & 0 & b^2+bx+x^2 \\ x^2 & 1 & x^3 \end{vmatrix}$$

$$\downarrow$$

$$\frac{1}{b}(a-x)(b-x)(a-b)\begin{vmatrix} b & 0 & a+b+x \\ b^2+bx & 0 & b^2+bx+x^2 \\ x^2 & 1 & x^3 \end{vmatrix}$$

$$= (a-x)(b-x)(a-b)[-(b^2+bx+x^2)+(a+b+x)+(b+x)]$$

$$= (a-x)(b-x)(a-b)[-b^2-bx-x^2+ab+b^2+bx++ax+bx+x^2]$$

$$= (a-x)(b-x)(a-b)[ab+ax+bx]$$

$$= (a-x)(b-x)(a-b)[(a+b)x+ab]$$

$x = a, \;\; x = b, \qquad x = \dfrac{-ab}{a+b}$

When $a = -b \quad a+b = 0 \;\; \Delta = ab(a-x)(b-x)(a-b)$

$\therefore 2^{nd}$ degree.